The impact of traditional risk measurement on the pro-cyclicality

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Motivation

- The capital computation based on risk estimation is usually done by financial institutions once a year and looked at in a **static way**, based on **past data**
- How well does the risk assessment (capital) hold in the future?
- Accepted idea: **risk measurements** made with 'regulatory' risk measures, are **pro-cyclical**
  - $\nearrow$ in times of crisis, they overestimate the future risk
  - $\searrow$ they underestimate it in quiet times

$\hookrightarrow$ We need to introduce **dynamics** in the measurement of risk, to be able to **quantify** this pro-cyclicality

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Two main goals

A - Quantifying the pro-cyclicality:

1. generalize in a simple way the static 'regulatory' risk measure VaR to a **dynamic one**
2. test the relevance and the **predictive power** of the SQP risk measure
3. **quantify empirically pro-cyclicality**

For this:

1. Consider the measurement itself as a stochastic process, introducing **Sample Quantile Process** (SQP) as a risk measure
2. (a) Play with the random measure defining the SQP

   (b) Define a **look-forward ratio** to see how the historical estimate of the SQP predicts the risk according to the volatility state
3. Use the **realized volatility** as a marker for the **market state**, ...
   ... analyzing the look-forward **SQP ratio conditioned** to the realized volatility

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B - Looking for explanations, we show that pro-cyclicality may be explained by two factors:

1. the very **way risk is measured**
2. **the clustering and return-to-the-mean of volatility**

For this:

1. Consider a simple **iid model** to show

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   ... empirically and theoretically

2. Use a simple **GARCH(1,1) model**

   ... to observe similar negative correlation

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Risk Measure: VaR

- In financial markets, most popular risk measure: **Value-at-Risk** (VaR)
- Given a loss random variable L (with cdf F_L), level $\alpha \in (0,1)$

$$\mathrm{VaR}(\alpha) = \inf\{\mathrm{x} \in \mathbb{R} : \mathbb{P}(\mathrm{L} \leq \mathrm{x}) \geq \alpha\} \underset{\text{strict.}\nearrow}{\overset{\mathrm{F_L}\text{cont.}}{=}} \mathrm{F_L^{-1}}(\alpha)$$

- Practically, VaR is estimated as an **empirical quantile**:
  Given a sample of n **historical losses** $(L_1, \cdots, L_n)$, $\alpha \in (0,1)$,

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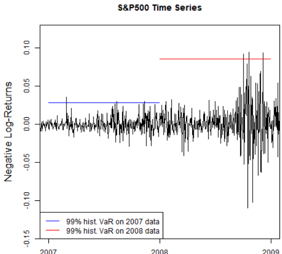

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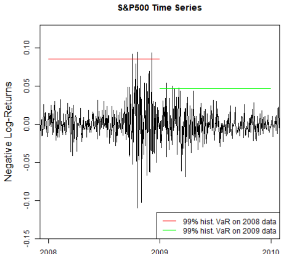

Dynamic extension of VaR: SQP

- **Sample Quantile Process** (SQP) (Miura (92), Akahori (95), Embrechts & Samorodnitsky (95)): Given $L = (L_t, t \geq 0)$, $\alpha \in (0,1)$, a fixed time frame T, and a random measure μ on $\mathbb{R}^+$, the SQP is defined at time t as

$$Q_{T,\alpha,t}(L) = \inf \left\{ x : \frac{1}{\int_{t-T}^{t} \mu(s)ds} \int_{t-T}^{t} \mathbb{I}_{(L_s \leq x)} \mu(s) ds \geq \alpha \right\}.$$

- Ex: $\mu =$ Lebesgue measure:
  the VaR process $(Q_{T,\alpha,t}(L))_t$ (with a rolling window T)

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Setup 1: Empirical Study

- Data: **11 stock indices**, daily log-returns from Jan. 1987 to Sept. 2018
- Dynamic 'rolling-window' VaR: $(Q_{T,\alpha,t}(L))_t$ denoted $(\text{VaR}_{T,\alpha,t}(L))_t$, with empirical estimator

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  - For simplicity: $T = 1y$, $\alpha = 95\%$, monthly rolling-window t

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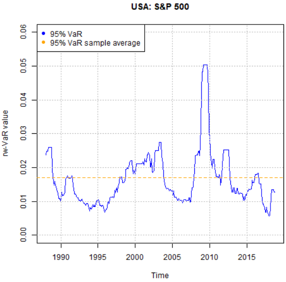

Setup 2 - Quality of risk prediction

- Introduce a **new quantity**: **look-forward ratio** of VaR's

$$R_{t,\alpha} = \frac{\widehat{VaR}_{1,\alpha,\mathbf{t+1y}}}{\widehat{VaR}_{T,\alpha,t}} \quad \text{with}$$

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Understanding the Dynamic Behavior

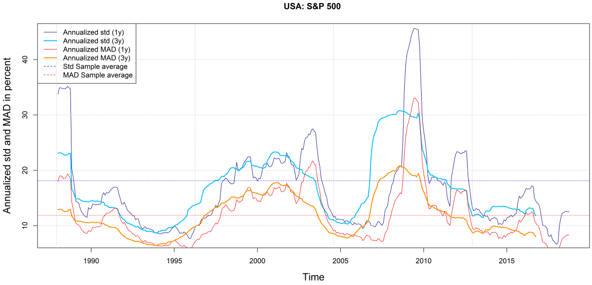
- Use a measure of annualized **realized volatility** as a proxy for **market states**

$$v_{k,n}(t-1) := \sqrt{252} \times \left\{ \frac{1}{n-1} \sum_{i=t-n}^{t-1} \left| X_i - \frac{1}{n} \sum_{i=t-n}^{t-1} X_i \right|^k \right\}^{1/k},$$

$k = 2$: $v_{2,n} = \hat{\sigma}(t)$ empirical standard deviation

$k = 1$: $v_{1,n} = \hat{\theta}(t)$ empirical mean absolute deviation (MAD)

- **Reasonable proxy** to discriminate between quiet and crisis periods



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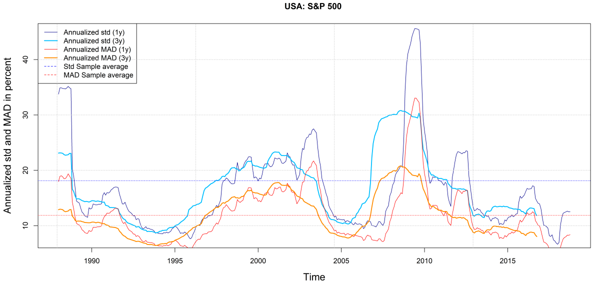

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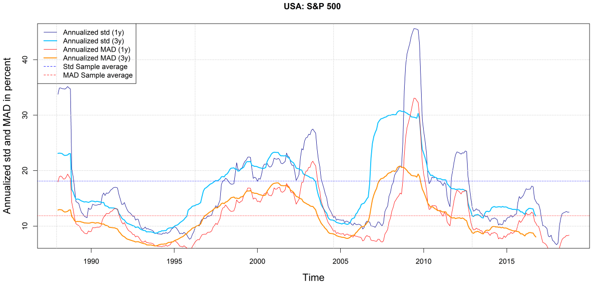

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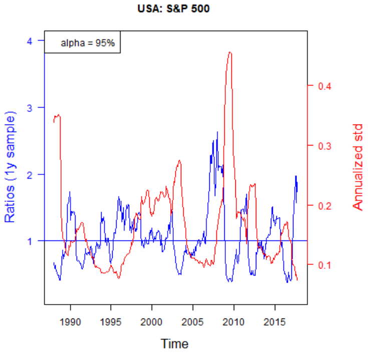
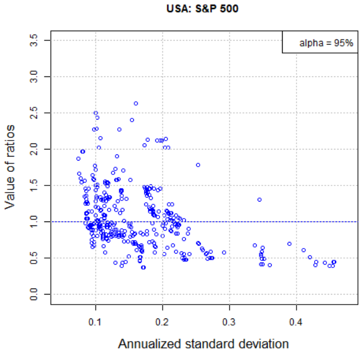
Relation between Volatility and VaR Ratios





- $\log(R_{T,\alpha,t})$ **negatively correlated** with annualized realized volatility:

Volatility year t	SQP ratio	Meaning
Low Volatility	High Ratio > 1	Underestimation of Risk
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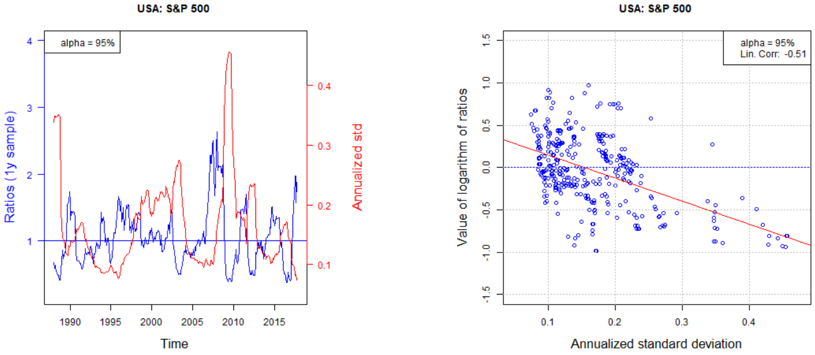

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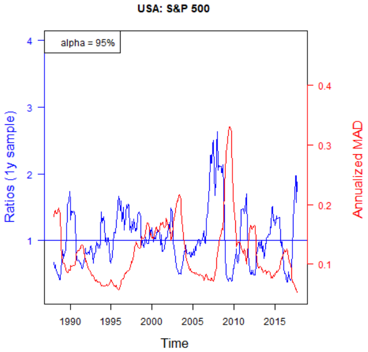

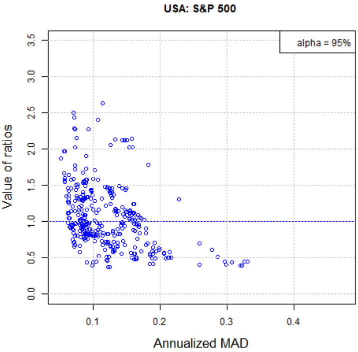

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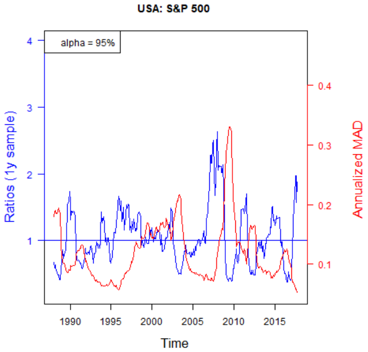

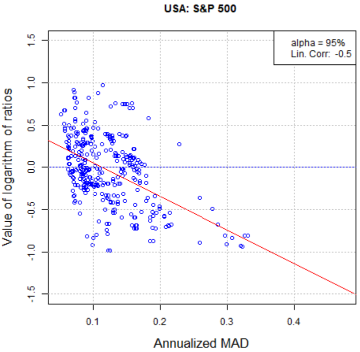

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Two factors that explain the pro-cyclicality

We are estimating empirically

$$Cor\left(\log\left|\frac{\widehat{VaR}_{t+1y}}{\widehat{VaR}_t}\right|, \hat{\sigma}_t\right) \text{ and } Cor\left(\log\left|\frac{\widehat{VaR}_{t+1y}}{\widehat{VaR}_t}\right|, \hat{\theta}_t\right)$$

- for an iid model
- for a GARCH(1,1) model
- using different underlying distributions

$\alpha = 95\%$ Model:	Data (average)	GARCH	iid
Correlation (log-ratios) with $\hat{\sigma}_t$	-0.54	-0.63	(-0.19)($t3$)/-0.40 (N)
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- Part of the pro-cyclicality would be intrinsically due to historic risk estimation ?
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X parent rv of an iid sample with mean μ, variance σ^2, quantile $q_X(p)$, $p \in (0,1)$

$m(X,r) = \mathbb{E}[|X-\mu|^r]$: measure of dispersion. Take $r = 1, 2$:
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h_i continuous real functions with existing derivatives h_i'
THEOREM: BIVARIATE CLT. *Under some conditions,*

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where the asymptotic covariance matrix $\Sigma^{(r)} = (\Sigma_{ij}^{(r)}, 1 \le i,j \le 2)$ *is well defined.*
For instance, in the Gaussian case (and $r = 2$*):*

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Conditions

Different conditions on X depending on the quantile estimator and the choice of measure of dispersion estimator:

Quantile Estimator	Asymptotic Normality	Joint asymptotics (with a measure of dispersion estimator)
$q_n(p)$	(H1): $0 < f_X(q_X(p)) < \infty$	(H1), (H3): $\begin{cases} F_X \text{ twice diff.able in nbhd. of } q_X(p), \\ F_X'' \text{bd. in that neighbourhood}, \\ F_X(q_X(p)) = p \end{cases}$

Measure of Dispersion Estimator	Asymptotic Normality	Joint asymptotics (with a quantile estimator)
$\hat{\sigma}_n^2$	(H2): $\begin{cases} \mathbb{E}[X^4] < \infty, \\ (X-\mu)^2 \text{ not constant} \end{cases}$	(H2)
$\hat{\theta}_n$	(Q1): $\begin{cases} \mathbb{E}[X^2] < \infty, \\ F_X \text{contin. at } \mu \end{cases}$	(Q1), (Q3): F_X Hölder-contin at μ

Conditions

Different conditions on X depending on the quantile estimator and the choice of measure of dispersion estimator:

Quantile Estimator	Asymptotic Normality	Joint asymptotics (with a measure of dispersion estimator)
$q_n(p)$	(H1): $0 < f_X(q_X(p)) < \infty$	(H1), (H3): $\begin{cases} F_X \text{ twice diff.able in nbhd. of } q_X(p), \\ F''_X \text{bd. in that neighbourhood}, \\ F_X(q_X(p)) = p \end{cases}$

Measure of Dispersion Estimator	Asymptotic Normality	Joint asymptotics (with a quantile estimator)
$\hat{\sigma}_n^2$	(H2): $\begin{cases} \mathbb{E}[X^4] < \infty, \\ (X-\mu)^2 \text{ not constant} \end{cases}$	(H2)
$\hat{\theta}_n$	(Q1): $\begin{cases} \mathbb{E}[X^2] < \infty, \\ F_X \text{contin. at } \mu \end{cases}$	(Q1), (Q3): F_X Hölder-contin at μ

A second factor: the clustering and return-to-the mean of volatility

- Use the **simplest version** of GARCH models, GARCH(1,1), to isolate the effect of clustering of volatility and its return to the mean
$$\sigma_t^2 = \omega + \alpha\, r_{t-1}^2 + \beta\, \sigma_{t-1}^2 \quad \text{with } r_{t+1} = \sigma_t\, \epsilon_t$$
where the innovation $\epsilon_t \in \mathcal{N}(0,1)$ or Student, to study the tail effect
- Fit the parameters ω, α, β to each full sample of the 11 indices, using a **robust optimization method** (Zumbach 2000) to obtain a **stationary solution** for the GARCH (s.t. $\alpha + \beta < 1$): the annualized volatility reproduces quite well the realized one (slightly higher)
- THEO: BIVARIATE FCLT. *Consider an augmented GARCH(*p*,* q*) process (Duan, 97). Introduce the vector* $T_{n,r}(X) = \begin{pmatrix} q_n(p) - q_X(p) \\ \hat{m}(X,n,r) - m(X,r) \end{pmatrix}$, $r \in \mathbb{Z}$. *Then, under some conditions, we have that, for* $t \in [0,1]$,
$$\sqrt{n}\, T_{[nt],r}(X) \overset{D_2[0,1]}{\rightarrow} \mathbf{W}_{\Gamma^{(r)}}(t) \quad \text{as } n \to \infty,$$
where $\mathbf{W}_{\Gamma^{(r)}}(t), t \in [0,1]$ *is the 2-dimensional Brownian motion with* $Cov(\mathbf{W}_{\Gamma^{(r)}}(t), \mathbf{W}_{\Gamma^{(r)}}(s)) = \min(s,t)\Gamma^{(r)}$, *with* $\Gamma^{(r)}$ *cov matrix well defined.*

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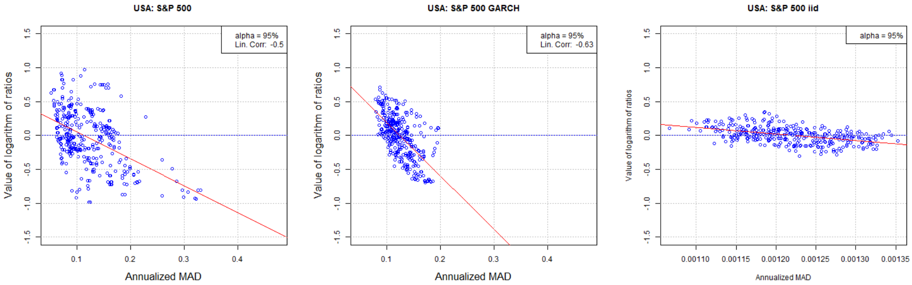
Comparing results



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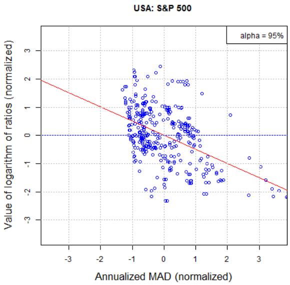

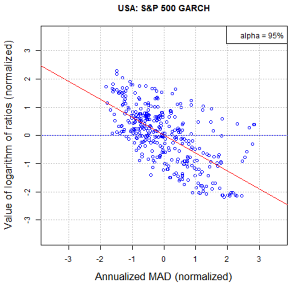

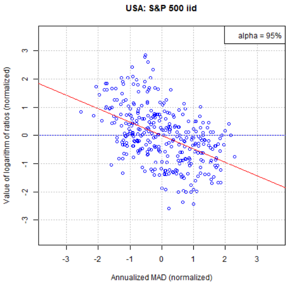

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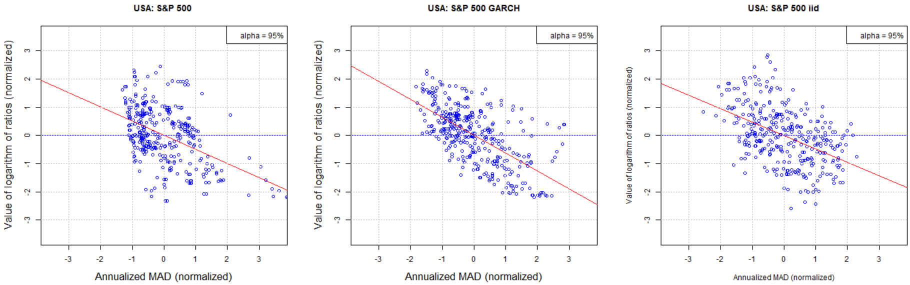

Model:	Data (S&P500)	GARCH	iid (Gaussian)
Correlation (log-ratios)	-0.50	-0.63	-0.34

Conclusion

- We introduced a 'dynamic generalization of VaR' - the **SQP**
- Pro-cyclicality of the SQP confirmed and **quantified** (by conditioning to realized volatility): During **high-volatility** periods, those risk measures **overestimate** the risks for the following years, whereas during **low-volatility** periods, they **underestimate** them
- Identification of 2 factors explaining pro-cyclicality of risk measurement, with a negative dependence between the realized volatility and the log SQP-ratios shown **empirically** and **theoretically**
  - (i) **the way risk is measured**, via iid model;
  - (ii) **clustering effect of the volatility**, via GARCH models
- Choice of dispersion measure matters: Think about **MAD** as a good alternative to Standard Deviation
- Ongoing work: the **design of the SQP** (the random measure μ) with the **proper dynamical behavior** as a good basis for **anti-cyclical** regulation

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Based on 3 preprints (2 submitted):

- M. Bräutigam, M. Dacorogna, M. Kratz (2018-19). Pro-Cyclicality of Traditional Risk Measurements: Quantifying and Highlighting Factors at its Source. *arXiv:1903.03969* (ESSEC WP1803)
- M. Bräutigam and M. Kratz (2019). On the Dependence between Quantile and Dispersion Estimators. *arXiv:1904.11871*.
- M. Bräutigam and Kratz, M. (2019). Bivariate FCLT for the Sample Quantile and Measures of Dispersion for Augmented GARCH(p, q) processes. (soon available)