The impact of traditional risk measurement on the pro-cyclicality

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INTRODUCTION

Motivation

- The capital computation based on risk estimation is usually done by financial institutions once a year and looked at in a static way, based on past data.
- How well does the risk assessment (capital) hold in the future?
- Accepted idea: risk measurements made with ‘regulatory’ risk measures, are pro-cyclical.
  - in times of crisis, they overestimate the future risk
  - they underestimate it in quiet times

→ We need to introduce dynamics in the measurement of risk, to be able to quantify this pro-cyclicality.

→ What are the factors that may explain this effect?
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A - Quantifying the pro-cyclicality:

1. generalize in a simple way the static 'regulatory' risk measure VaR to a dynamic one
2. test the relevance and the predictive power of the SQP risk measure
3. quantify empirically pro-cyclicality

For this:

1. Consider the measurement itself as a stochastic process, introducing Sample Quantile Process (SQP) as a risk measure
2. (a) Play with the random measure defining the SQP
   (b) Define a look-forward ratio to see how the historical estimate of the SQP predicts the risk according to the volatility state
3. Use the realized volatility as a marker for the market state, ... analyzing the look-forward SQP ratio conditioned to the realized volatility
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2. Use a simple GARCH(1,1) model
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Risk Measure: VaR

- In financial markets, most popular risk measure: **Value-at-Risk (VaR)**
- Given a loss random variable $L$ (with cdf $F_L$), level $\alpha \in (0, 1)$

$$\text{VaR}(\alpha) = \inf\{x \in \mathbb{R} : \mathbb{P}(L \leq x) \geq \alpha\} \overset{F_L\text{cont.}}{=} F_L^{-1}(\alpha)$$

- Practically, VaR is estimated as an **empirical quantile**: Given a sample of $n$ historical losses $(L_1, \cdots, L_n)$, $\alpha \in (0, 1)$,

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![S&P500 Time Series](image)
Dynamic extension of VaR: SQP

- **Sample Quantile Process (SQP) (Miura (92), Akahori (95), Embrechts & Samorodnitsky (95))**: Given $L = (L_t, t \geq 0)$, $\alpha \in (0, 1)$, a fixed time frame $T$, and a random measure $\mu$ on $\mathbb{R}^+$, the SQP is defined at time $t$ as

$$Q_{T,\alpha,t}(L) = \inf \left\{ x : \frac{1}{\int_{t-T}^{t} \mu(s)ds} \int_{t-T}^{t} \mathbb{I}(L_s \leq x) \mu(s)ds \geq \alpha \right\}.$$

- **Ex**: $\mu = \text{Lebesgue measure}$:
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- Data: **11 stock indices**, daily log-returns from Jan. 1987 to Sept. 2018
- Dynamic ‘rolling-window’ VaR: \((Q_{T,\alpha,t}(L))_t\) denoted \((\hat{\text{VaR}}_{T,\alpha,t}(L))_t\), with empirical estimator

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- For simplicity: \(T = 1\) year, \(\alpha = 95\%\), monthly rolling-window \(t\)
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- For simplicity: \( T = 1y \), \( \alpha = 95\% \), monthly rolling-window \( t \)
Setup 2 - Quality of risk prediction

- Introduce a new quantity: look-forward ratio of VaR’s

\[ R_{t,\alpha} = \frac{\hat{VaR}_{1,\alpha,t+1y}}{\hat{VaR}_{T,\alpha,t}} \]

\( \hat{VaR}_{T,\alpha,t} \) used as a predictor of the risk 1 year later \( (t + 1y) \)

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- Use a measure of annualized realized volatility as a proxy for market states

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- Reasonable proxy to discriminate between quiet and crisis periods
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\[ \log(R_{T,\alpha,t}) \text{ negatively correlated} \] with annualized realized volatility:

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Two factors that explain the pro-cyclicality

We are estimating empirically

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\text{Cor} \left( \log \left( \frac{\hat{VaR}_{t+1}}{VaR_t} \right), \hat{\sigma}_t \right) \quad \text{and} \quad \text{Cor} \left( \log \left( \frac{\hat{VaR}_{t+1}}{VaR_t} \right), \hat{\theta}_t \right)
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- for an iid model
- for a GARCH(1,1) model
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A first factor: the way risk is measured

\( X \) parent rv of an iid sample with mean \( \mu \), variance \( \sigma^2 \), quantile \( q_X(p), p \in (0, 1) \)

\[ m(X, r) = \mathbb{E}[|X - \mu|^r] \] measure of dispersion. Take \( r = 1, 2 \):

\( m(X, 2) = \sigma^2 \) and \( m(X, 1) = \) Mean Absolute Deviation (MAD).

Consider their empirical estimators

\[
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\( h_i \) continuous real functions with existing derivatives \( h'_i \)

**Theorem: Bivariate CLT.** Under some conditions,

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where the asymptotic covariance matrix \( \Sigma^{(r)} = (\Sigma_{ij}^{(r)}, 1 \leq i, j \leq 2) \) is well defined.

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- Use the **simplest version** of GARCH models, GARCH(1,1), to isolate the effect of clustering of volatility and its return to the mean

\[ \sigma^2_t = \omega + \alpha r^2_{t-1} + \beta \sigma^2_{t-1} \quad \text{with} \quad r_{t+1} = \sigma_t \epsilon_t \]

where the innovation \( \epsilon_t \in \mathcal{N}(0,1) \) or Student, to study the tail effect

- Fit the parameters \( \omega, \alpha, \beta \) to each full sample of the 11 indices, using a **robust optimization method** (Zumbach 2000) to obtain a **stationary solution** for the GARCH (s.t. \( \alpha + \beta < 1 \)): the annualized volatility reproduces quite well the realized one (slightly higher)

**Theo: Bivariate FCLT.** Consider an augmented GARCH\((p, q)\) process (Duan, 97). Introduce the vector \( T_{n,r}(X) = \left( \begin{array}{c} q_n(p) - q_x(p) \\ \hat{m}(X, n, r) - m(X, r) \end{array} \right), \ r \in \mathbb{Z}. \)

Then, under some conditions, we have that, for \( t \in [0, 1], \)

\[ \sqrt{n} T_{[nt], r}(X) \overset{D}{\underset{t \to 1}[0,1]}{\to} \mathbf{W}_{\Gamma(r)}(t) \quad \text{as} \quad n \to \infty, \]

where \( \mathbf{W}_{\Gamma(r)}(t), t \in [0, 1] \) is the 2-dimensional Brownian motion with \( \text{Cov}(\mathbf{W}_{\Gamma(r)}(t), \mathbf{W}_{\Gamma(r)}(s)) = \min(s, t) \Gamma(r) \), with \( \Gamma(r) \) cov matrix well defined.
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Comparing results

USA: S&P 500

USA: S&P 500 GARCH

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Comparing results

EXPLAIN

Model: Data (S&P500) GARCH iid (Gaussian)

Correlation (log-ratios) -0.50 -0.63 -0.34
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- We introduced a ‘dynamic generalization of VaR’ - the SQP

- Pro-cyclicality of the SQP confirmed and quantified (by conditioning to realized volatility): During high-volatility periods, those risk measures overestimate the risks for the following years, whereas during low-volatility periods, they underestimate them.

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Based on 3 preprints (2 submitted):


- **M. Bräutigam and Kratz, M.** (2019). Bivariate FCLT for the Sample Quantile and Measures of Dispersion for Augmented GARCH\((p, q)\) processes. (soon available)