Subjective state space under radical uncertainty

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This talk is about decision theory under radical uncertainty, namely, asking how do, and how should, people make decisions in situations of radical uncertainty. You probably already know the standard answer under risk and uncertainty in economics, namely, that people should maximize (subjective) expected utility. I don't think that this is a bad answer if we know what "utility" and objective (subjective) "probability" mean.

Risk and uncertainty

Risk refers to circumstances with known probabilities, as opposed to situations of uncertainty, where probabilities are not known. There are two urns, each containing 100 balls. Urn I contains 50 red balls and 50 black balls. Urn II contains 100 balls, each of which is known to be either red or black, but you have no information about how many of the balls are red and how many are black.

- Urn I describes a risky situation,
- Urn II describes an uncertain situation.

Risk and uncertainty

Under risk, alternative are lotteries, a list of outcomes associated with known probabilities.

- you win 100 euros if a red ball is drawn from urn I and nothing if a black ball is drawn.

Under uncertainty, alternative are acts, a list of outcomes associated with known events.

- you win 100 euros if a red ball is drawn from urn II and nothing if a black ball is drawn.

Example of classical decision criterion

- Under risk, for a lottery $l = (x_1, p_1; \ldots; x_n, p_n)$, the expected utility is given by
$$\sum_i p_i u(x_i).$$
- Under uncertainty, for an act $f = (x_1, E_1; \ldots; x_n, E_n)$, the subjective expected utility is given by
$$\sum_i \pi(E_i) u(x_i).$$

Radical uncertainty

Consider a terminally ill patient whose doctor suggests two treatments. The first is an established pharmaceutical. Numerous published studies concur that this drug is successful in thirty percent of cases. The second is a new experimental surgery. Its preliminary trials suggest a success rate between twenty and forty-five percent. The two treatments are mutually exclusive, so the patient must choose between them. Can we help the patient by framing her problem with either a theory of choice under risk or a theory of choice under uncertainty.

Radical uncertainty

We cannot frame the patient's problem with a risk framework; the surgery is associated with an ambiguous range of possible success rates. Also, to invoke the uncertainty machinery, our patient must be able to: first, determine the relevant events; second, decide how each choice assigns consequences to these events.

- Regarding the events which occur during a surgery, the patient does not understand what the relevant events are.
- She knows only the information presented by her physician, expressed entirely in the space of probabilities over consequences.
- Even given a comprehensive list of events, the designers of the experimental surgery are unsure which events would make the surgery more likely to be successful.

Studies are needed exactly because the mapping from events to outcomes that actually represents this new medical procedure is still unknown.

There seem to be many situations where the decision maker cannot properly conceptualize the events and how to events are associated with outcomes , but still has to decide on a course of action.

- From risk to uncertainty, you replace objective probabilities by subjective probabilities.
- We seek, under radical uncertainty to derive, from choice, subjective events to apply known theories of classical uncertainty

Our decision maker will behave "as if" he have a representation of the events in her mind.

Even without understanding the states underlying her choices, the decision maker may still understand how her choices affect consequences, which are the ultimate objects of her utility. She can understand how an option might make her feel, without understanding the causal mechanism or act that delivers that feeling. Her preferences must reflect a subjective (hypothetic) set of events. Such behaviors seem especially plausible when summary information is given to the decision maker by an expert, like the doctor in our example.

Preliminaries

- X a finite nonempty consequence space.
- $\Delta(X)$ is the set of all lotteries over X.
- A nonempty, compact, convex subset of $\Delta(X)$ is interpreted as a description of all outcomes of an alternative and denoted K (or L, M, etc.). The set of alternatives is denoted $\mathcal{K}$
- A menu is a set of alternative, denoted $\mathcal{A}$, $\mathcal{B}, \ldots$. The set of menu is denoted $\mathcal{F}$

- The pharmaceutical is a lottery: $(\text{life}, 30\%; \text{death}, 70\%)$
- The surgery is an alternative: set of lotteries $\{(\text{life}, p\%; \text{death}, (100 - p)\%) \mid p \in [20, 45]\}$
- pharmaceutical or surgery is a menu.

We have in mind an environment where the DM chooses a menu (set of alternatives) and at a later stage the nature will choose an element in each alternatives of the menu. After the move of the nature, the DM will choose among the resulting menu of lotteries, but we do not explicitly model this second choice. To clarify, we refer to the preference $\succsim$ over menus of alternatives as the ex ante preference of the DM.

Hypothesis

W.O $\succsim$ is complete and transitive.

Mon. For all $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, if $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{A} \succsim \mathcal{B}$.

W.Ind For all $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, if $\mathcal{B} \subseteq \mathcal{A}$, then for all $\mathcal{C} \in \mathcal{F}$ and all $\alpha \in (0, 1]$,

$$\mathcal{A} \sim \mathcal{B} \Rightarrow \alpha\mathcal{A} + (1-\alpha)\mathcal{C} \sim \alpha\mathcal{B} + (1-\alpha)\mathcal{C}$$
$$\mathcal{A} \succ \mathcal{B} \Rightarrow \alpha\mathcal{A} + (1-\alpha)\mathcal{C} \succ \alpha\mathcal{B} + (1-\alpha)\mathcal{C}$$

Dom. For all $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, if for every $a \in A \in \mathcal{A}$ and $b \in B \in \mathcal{B}$, $a \succsim b$, then $\mathcal{A} \succsim \mathcal{A} \cup \mathcal{B}$

Result

A representation of $\succsim$ is a quadruple (S, u, ϕ, I), consisting of a state space S, an affine utility function $u : \Delta(X) \to \mathbb{R}$, an affine map $\phi : \mathcal{K} \to C(S, \Delta(X))$ which maps alternative to subjective act and a monotonic, normalized, continuous aggregator $I : C(S, u(\mathcal{M}(X))) \to \mathbb{R}$. such that (i) for every $K \in \mathcal{K}$, for every $s \in S$, $\phi(K)(s) \in K$, (ii) S is minimal and unique, and (iii) for all $\mathcal{A}, \mathcal{B} \in \mathcal{F}$

$$\mathcal{A} \succsim \mathcal{B} \Leftrightarrow I\left(\left(\max_{K \in \mathcal{A}}(u \circ \phi(K)(s))\right)_{s \in S}\right) \geq I\left(\left(\max_{K \in \mathcal{B}}(u \circ \phi(K)(s))\right)_{s \in S}\right)$$

Under few assumption on the preferences of the DM, she behaves "as if" there exists a (subjective) set of state of the world (elementary events). This subjective state space is unique, each alternative is mapped to a subjective act (a function which maps event to consequence) and the utility is independent of the state.

- In other words, radical uncertainty is reduced to subjective uncertainty.

For a simple alternative K, if the aggregator I is linear it is possible to have a subjective expected utility given by

$$\int_S u \circ \phi(K)(s) d\pi(s).$$

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If you compare with the expected utility criterion, an alternative (set of consequence) is transformed into an act and its evaluation is achieved by a subjective expected utility criterion.

- Decision making under radical uncertainty can be reduced to classical uncertainty.
- Derivation from preferences of a unique and minimal subjective state space.